Differential Calculus [Indeterminate form and L. Hospital's rule]

Let us consider on two limits

$$\underset{x \to 2}{Lt} \frac{x^2 - 4}{x - 2} \quad \text{and} \quad \underset{x \to \infty}{Lt} \frac{2x^2 - 3x}{1 - 5x^2}$$

In the first limit if we put $x = 2$ we get $0/0$ and in second if we put ∞, we get $\infty/-\infty$. Both of these are called indeterminate forms. In both of these cases there are competing interests or rules and it's not clear which will apply.

In the case of $0/0$ we typically think of a fraction that has a numerator of zero as being zero. However, we also tend to think of fractions in which the denominator is zero, in the limit, as infinity or not exist at all. Likewise, we tend to think of a fraction in which the numerator and denominator are the same as one. So, which will win out ? or will neither win out and they all 'cancel out' and the limit will reach some other value.

In the case of $\infty/-\infty$ we have similar set of problems. If the numerator of a fraction is going to infinity we tend to think of the whole fraction going to infinity. Also, if the denominator is going to infinity, we tend to think of the fraction as going to zero. We also have the case of a fraction in which the numerator and denominator are the same (ignoring the minus sign) and so we get -1. Again, it's not clear which of these will win out, if any of them will win out.

This is the problem with indeterminate forms. It's just not clear what is happening in the limit. There are other types of indeterminate forms as well. Some other types are,

$$(0)(\pm\infty) \quad (1^{\infty}) \quad (0^0) \quad (\infty^0) \quad (\infty-\infty)$$

These all have competing interests or rules that should happen and it's just not clear which, if any, of the rules will win out. Here we discuss how to deal with these kind of limits.

For the two limits we work them as follows.

$$\lim_{x \to 2} \frac{x^2-4}{x-2} = \underset{x \to 2}{\text{Lt}} \frac{(x+2)(x-2)}{(x-2)} = \underset{x \to 2}{\text{Lt}} (x+2) = 2+2 = 4.$$

and
$$\underset{x \to \infty}{\text{Lt}} \frac{2x^2-3x}{1-5x^2} = \underset{x \to \infty}{\text{Lt}} \frac{2-\frac{3}{x}}{\frac{1}{x^2}-5} = -\frac{2}{5}$$

In the first limit we simply factored, canceled and took the limit and in the second we factored out an x^2 from both the numerator and denominator and took the limit.

So, we can deal with some of these. However, what about the following two limits.

$$\underset{x \to 0}{\text{Lt}} \frac{\sin x}{x} \quad \text{and} \quad \underset{x \to \infty}{\text{Lt}} \frac{e^x}{x^2}$$

This first is a $0/0$ indeterminate form, but we can't factor this one, The second is an ∞/∞ indeterminate form, but we can't factor an x^2 out of the numerator. So, nothing we've got of tricks will work with these two limits.

This is where the subject of this section comes into play.

L. Hospital's Rule :- Suppose that we have one of the following cases,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \quad \text{or} \quad \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$$

Where a can be any real number, infinity or negative infinity. In these cases we have,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

So, L. Hospital's rule tells that if an indeterminate form $0/0$ or ∞/∞ all need to do is differentiate the numerator and differentiate the denominator and then take the limit.

Ex. 1. Evaluate the limit $\lim_{x \to 0} \frac{\sin x}{x}$

Soln. Here we saw that it is a $0/0$ indeterminate form so let just apply L. Hospital's rule

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \frac{\cos 0}{1} = \frac{1}{1}$$

$$= 1.$$

Ans